Definition of a Derivative

For a function $f(x)$, its derivative can be defined using

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Linearity

The derivative of the sum of functions is the sum of their derivatives

$$\begin{aligned}
[f(x) + g(x)]' &= \lim_{h \to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} \\
&= \lim_{h \to 0} \left\{ \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right\} \\
&= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x) + g'(x)
\end{aligned}$$

We can also pull constants out of a derivative

$$\begin{aligned}
[cf(x)]' &= \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h} \\
&= c \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= cf'(x)
\end{aligned}$$

Chain Rule

Derivatives behave 'like' fractions where

$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

In Leibniz notation, we can write

$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$

To derive this, first define $k = g(x+h) - g(x)$. Then,

$$\begin{aligned}
[f(g(x))]' &= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} \\
&= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} \cdot \frac{g(x+h) - g(x)}{g(x+h) - g(x)} \\
&= \lim_{h,k \to 0} \frac{f(g(x) + k) - f(g(x))}{h} \cdot \frac{g(x+h) - g(x)}{k} \\
&= \lim_{k \to 0} \frac{f(g(x) + k) - f(g(x))}{k} \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(g(x)) \cdot g'(x)
\end{aligned}$$

Product Rule

Below is a non-rigorous visual proof for why

$$[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$